

# Non-holonomic Control II : Non-holonomic Quantum Devices

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### **Abstract**

In this paper, we show how the non-holonomic control technique can be employed to build completely controlled quantum devices. Examples of such controlled structures are provided.

## **1 Introduction**

In the previous article, we showed that quantum systems become “non-holonomic” when perturbed in a certain time-dependent way : as a result of the perturbation, all global constraints on the dynamics are removed and the system becomes fully controlled. The straightforward application of the non-holonomic control to groups of interacting two-level systems, the so-called qubits, is a promising way to achieve computational algorithms widely discussed in quantum informatics [1, 2, 3, 4]. Indeed, quantum computations [5] are achieved through combining well-chosen quantum gates [6], which are the analog of the classical logic gates and consist in particular evolutions of the system chosen as a computer. In this context, the non-holonomic control is a precious means for implementing any gate and thus performing any computation on an arbitrary quantum system. However, the direct control of an  $N$ -qubit computer’s evolution is a very heavy (or even intractable) computational task when  $N$  is a large number, since it requires the control of  $4^N$  physical parameters. To overcome this impediment one can combine completely controlled cells in a manner which depends on the kind of the problem to be solved : the adaptation of the structure of the quantum device, obtained as a particular arrangement of controlled elements, to the computation to be performed allows one to decrease the number of the free control parameters needed, and thus the complexity of the control problem. The complete control of the whole compound device is thus assured by the controllability of the individual cells as well as their connections with each other within the architecture of the computer.

In the first section of this paper, we provide an example of a unit cell, which is completely controlled through non-holonomic interactions. In the second section, we propose two different devices composed of such cells, the arrangements of which suit particularly well universal quantum computations and simulation of quantum field dynamics, respectively. In the third section, we finally describe a toy device that can perform quantum computations on 9 qubits and show in particular how it can perform the discrete Fourier transform on 9 qubits.

## **2 Completely Controlled Unit Cell**

One way to construct a completely controlled but not immediately universal quantum device is to build it up from small parts, called “unit cells”, each of which is non-holonomic and therefore directly and universally controllable. The

proper functioning of the device relies then on the appropriate connection of the cells. In this way the universality of the device is obtained indirectly, not by applying a huge number of controls, but by smartly connecting the cells and choosing the sequence of operations performed. There is no general prescription on how to construct a particular device; this requires expertise in the art of “programming” the operations of the cells and their interactions.

## 2.1 Cell structure

Fig. 1 shows an example of a completely controlled unit cell composed of three two-level atoms, each with ground and excited states  $|0\rangle$  and  $|1\rangle$ , having distinct transition frequencies  $\omega_1^a$ ,  $\omega_2^a$ , and  $\omega_3^a$ . The atoms are subject to dipole-dipole interactions and are coupled to two external fields: an electromagnetic field  $E_\omega = \mathcal{E}_\omega \cos \omega t$  of nearly resonant frequency  $\omega$ , and a static electric field  $E_S$ . The dipole-dipole interaction is fixed and determines the principal, unperturbed Hamiltonian of the system,  $\hat{H}_0$ , while the external fields provide two controllable perturbations,  $\hat{P}_\omega$  and  $\hat{P}_S$ . The Hilbert space of the system has a “computational basis” of  $N = 2^3 = 8$  states,  $|x\rangle \equiv |x_2 x_1 x_0\rangle \equiv |x_2\rangle |x_1\rangle |x_0\rangle$ ,  $x = 0, 1, \dots, 7$ , where the state of the  $i$ th atom encodes the  $i$ th binary digit of  $x = \sum_{r=0}^2 x_r 2^r$  as a qubit [see Fig. 1(b)]. The crucial requirement is the non-holonomic character of the interaction :  $\hat{H}_0$ ,  $\hat{P}_\omega$ ,  $\hat{P}_S$ , and their commutators of all orders must span the linear space of  $8 \times 8$  Hermitian matrices. This is indeed the case for the system shown in Fig. 1, for which principal Hamiltonian and perturbations are given, in the computational basis and assuming resonant approximation, by the

matrices

$$\hat{H}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_1 & D_{12} & 0 & D_{13} & 0 & 0 & 0 \\ 0 & D_{21} & A_2 & 0 & D_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{12} & 0 & D_{23} & D_{13} & 0 \\ 0 & D_{31} & D_{32} & 0 & A_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{32} & 0 & A_{13} & D_{21} & 0 \\ 0 & 0 & 0 & D_{31} & 0 & D_{12} & A_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_\sigma \end{pmatrix}, \quad (1)$$

$$C_\omega \hat{P}_\omega = \begin{pmatrix} 0 & V_1 & V_2 & 0 & V_3 & 0 & 0 & 0 \\ V_1 & 0 & 0 & V_2 & 0 & V_3 & 0 & 0 \\ V_2 & 0 & 0 & V_1 & 0 & 0 & V_3 & 0 \\ 0 & V_2 & V_1 & 0 & 0 & 0 & 0 & V_3 \\ V_3 & 0 & 0 & 0 & 0 & V_1 & V_2 & 0 \\ 0 & V_3 & 0 & 0 & V_1 & 0 & 0 & V_2 \\ 0 & 0 & V_3 & 0 & V_2 & 0 & 0 & V_1 \\ 0 & 0 & 0 & V_3 & 0 & V_2 & V_1 & 0 \end{pmatrix}, \quad (2)$$

$$C_S \hat{P}_S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta_\sigma \end{pmatrix}. \quad (3)$$

Here  $D_{ij} = d_i d_j / R_{ij}^3$  denotes the dipole-dipole coupling of the  $i$ th and  $j$ th atoms at distance  $R_{ij}$ , with  $d_i$  the  $i$ th atom dipole matrix element, and  $V_i = \mathcal{E}_\omega d_i$  is the dipole coupling of the  $i$ th atom to the external electromagnetic field. The excitation energy detunings of single atoms  $A_i = \hbar(\omega_i^a - \omega)$  determine the detunings of pairs of atoms  $A_{ij} = A_i + A_j$  and the total detuning  $A_\sigma = A_1 + A_2 + A_3$ . Their values can be changed by variation of a static electric field  $E_S$  (Stark effect), which results in energy shifts  $A_i \rightarrow A_i + \Delta_i$  for single atoms, where  $\Delta_i = \alpha_i E_S$  depend on atom-specific electric permeability constants  $\alpha_i$ , and similar shifts  $\Delta_{ij} = \Delta_i + \Delta_j$  and  $\Delta_\sigma = \Delta_1 + \Delta_2 + \Delta_3$  for two and three atomic detunings respectively.

Note that, by a proper choice of  $\Delta_i$  and  $\omega$ , one can set two of three  $A_i$  to zero. Moreover, for clarity we also set to zero the third  $A_i$ , which remains just a part of  $\hat{H}_0$  otherwise. Hence, hereafter, all  $\Delta_i$  denote just the deviations from zero resulting from the variation  $E_S$  of the Stark field. The last together with the amplitude  $= \mathcal{E}_\omega$  serve as a time dependent control parameters  $C_S$  and  $C_\omega$  respectively. The matrices  $\hat{P}_S$  and  $\hat{P}_\omega$  contain therefore only the permeabilities  $\alpha_i$  and the dipole moments  $d_i$  respectively.

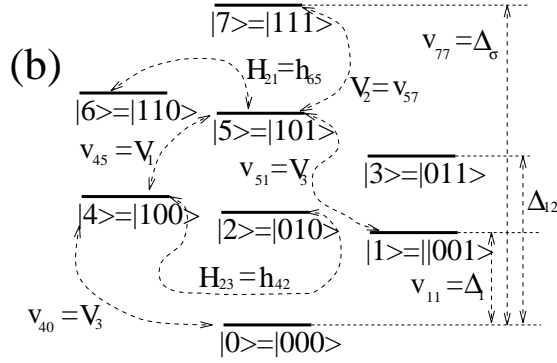
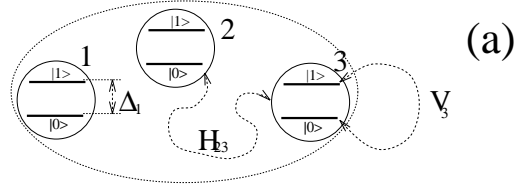


Figure 1: Realization of a unit cell: A compound system of three two-level atoms interacting with external electromagnetic and static electric fields. (a) The  $i$ th atom has ground and excited states  $|0\rangle_i$  and  $|1\rangle_i$  with excitation energy  $A_i + \Delta_i$  that can be modified by the static field; transition amplitude in the electromagnetic field is  $V_i$ ; the dipole-dipole coupling of the  $i$ th and  $j$ th atoms is  $D_{ij}$ . (b) The computational basis states and their relation to matrix elements of the principal Hamiltonian  $\hat{H}_0$  and the perturbations  $\hat{P}_\omega$  and  $\hat{P}_S$  of Eqs. (1-3).

## 2.2 Cell control

To exert direct universal control over the unit cell we employ the non-holonomic control technique presented in the previous article. (i) We fix  $N^2 = 64$  consecutive time intervals of equal length  $\tau = \frac{T}{64}$ , during which the two perturbations are alternately applied to the system : in the  $k$ th interval the perturbation is  $\hat{P}_k = \hat{P}_S$  for odd  $k$  and  $\hat{P}_k = \hat{P}_\omega$  for even  $k$ , where  $k = 1, 2, \dots, 64$ . The strength of  $\hat{P}_k$  is denoted by  $C_k$ , and corresponds either to  $\mathcal{E}_\omega$  or  $E_S$ , depending on the parity of  $k$ . Thus, the evolution of the system is governed by a Hamiltonian which is constant on each interval:

$$\hat{H}(t) = \hat{H}_0 + C_k \hat{P}_k \quad t \in [(k-1)\tau, k\tau]. \quad (4)$$

(ii) We look for the 64-dimensional vector  $\vec{C}^{(0)}$  such that

$$\hat{U}(\vec{C}^{(0)}) \equiv \prod_{k=1}^{64} \exp \left[ -\frac{i}{\hbar} (\hat{H}_0 + C_k^{(0)} \hat{P}_k) \tau \right] = \hat{I}. \quad (5)$$

To this end, we first solve the “8th root” of Eq. (5),

$$\prod_{k=1}^8 \exp \left[ -\frac{i}{\hbar} (\hat{H}_0 + c_k \hat{P}_k) \tau \right] = \hat{I}^{1/8}, \quad (6)$$

by minimizing the functional  $\sum_{j=0}^8 |a_j(\{c_k\}_{k=1\dots 8})|^2$  to 2, where  $\{a_j\}$  denote the coefficients of the characteristic polynomial of the matrix product in Eq.(6). This provides a sequence of eight values,  $c_1, c_2, \dots, c_8$ , the repetition of which yields the desired vector  $\vec{C}^{(0)}$ . (iii) Finally, we compute the required vector  $\vec{C}$  as described in the previous article. If the target evolution  $\hat{U}_{arbitrary} = \hat{U}_\epsilon \equiv \exp(-i\hat{\mathcal{H}}\epsilon)$  is close to the identity (*i.e.*  $\epsilon$  is small), one determines the variations  $\delta C_k$  to first order in  $\epsilon$  by solving the linear equations

$$\sum_{k=1}^{64} \frac{\partial \hat{U}}{\partial C_k}(\vec{C}^{(0)}) \delta C_k = -i\hat{\mathcal{H}}\epsilon, \quad (7)$$

and, replacing  $\vec{C}^{(0)}$  by  $\vec{C}^{(0)} + \delta \vec{C}$ , one repeats the same operation, and so on, until one gets the vector  $\vec{C}$  which checks  $\hat{U}(\vec{C}) = \hat{U}_\epsilon$  with desired accuracy. To perform an arbitrary unitary transformation  $\hat{U}_{arbitrary} = \hat{U}_\epsilon$ , with  $\epsilon$  taking any value in  $[0, 2\pi]$  and not necessarily small, we divide the work into “small” steps : we apply the transformation  $\hat{U}(\vec{C}^{(\frac{1}{n^*})}) = \hat{U}_{\epsilon/n^*} = \left(\hat{U}_{arbitrary}\right)^{\frac{1}{n^*}}$  repeatedly  $n^*$  times, with  $n^*$  determined as described in the previous article, and obtain

$$\left[ \hat{U}(\vec{C}^{(\frac{1}{n^*})}) \right]^{n^*} = \left[ \hat{U}_{\epsilon/n^*} \right]^{n^*} = \left[ \left( \hat{U}_{arbitrary} \right)^{\frac{1}{n^*}} \right]^{n^*} = \hat{U}_{arbitrary}. \quad (8)$$

In Fig. 2 we show examples of unit cell control, where appropriately chosen parameters  $C_k^{(0)}$  and variations  $\delta C_k$  achieve unitary transformations on the unit cell: the Toffoli-gate transformation (see Appendix A), two-qubit permutations  $\hat{p}_{ij}|a\rangle_i|b\rangle_j = |b\rangle_i|a\rangle_j$  ( $a, b = 0, 1$ ), and the conditional phase shift employed in the quantum discrete Fourier transform (discussed in Sec. 4). The transformation is achieved either directly ( $n^* = 1$ ) or by 8 repetitions ( $n^* = 8$ ). The operators  $\hat{H}_0$ ,  $\hat{P}_\omega$  and  $\hat{P}_S$  are chosen with arbitrary realistic values. We take  $D_{12} = 1.1E_u$ ,  $D_{23} = 0.946E_u$ ,  $D_{13} = 0.86E_u$ , and  $T = 250\hbar/E_u$ , where  $E_u \sim 10^{-18}$  erg is the typical energy scale. For odd  $k$  we switch off the external electromagnetic field,  $V_{1;2;3} = 0$ , and tune the atomic excitation energies by the Stark field  $E_S$  such that  $\Delta_{1;2;3} = (0.1; 0.11; 0.312)E_u$ . For even  $k$  we set  $E_S = 0$ , that is  $\Delta_{1;2;3} = 0$ , and take  $V_{1;2;3} = (0.3; 0.33; 0.24)E_u$ .

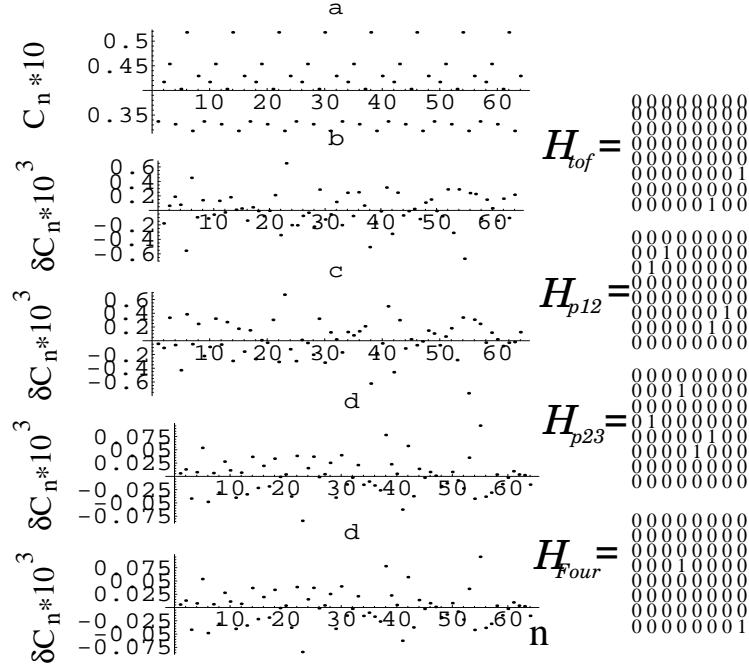


Figure 2: (a) Control parameters  $C_k^0$  for the identity transformation  $\hat{I}$ . Variations  $\delta C_k$  achieving the transformation  $\hat{U}_{\epsilon/8}$  on the cell, with  $\hat{U}_\epsilon \equiv (\hat{U}_{\epsilon/8})^8$  equal to: (b) the permutation  $\hat{U}_{p12}$ ; (c) the permutation  $\hat{U}_{p23}$ ; (d) the Toffoli-gate transformation  $\hat{U}_{Toff}$ . (e) Variations  $\delta C_k$  achieving the conditional phase shift  $\hat{B}(\phi) = \exp(-i\phi\hat{H}_B)$ , at  $\phi = \pi/32$ , employed in the quantum discrete Fourier transform.

### 3 Completely Controlled Quantum Devices

Once completely controlled unit cells have been constructed, a compound device can be assembled from such elements in an architecture which depends on the specific function it has to perform. Fig. 3 shows two possible arrangements of unit cells designed for two different purposes : the first one suites more the purpose of quantum computing, while the second is more useful for simulating lattice quantum field dynamics.

The first device (Fig. 3(a)) is organized in a tree-like structure. In this arrangement, the quantum state of one atom in each cell can be exchanged with the state of an atom in the closest parent joint of the tree. Even though the simplest way to make the exchange is to straightforwardly displace the atom to the parent joint, the exchange or transport of the state without moving the atom can be more practical. The tree-like architecture and the possibility to perform all the unitary transformations (including all the permutations) in each unit cell allow one to put together the states of any three two-level atoms of the device and make them interfere after at most  $s = 6 \log_3 n$  state exchanges, by moving them toward the root of the tree to a common cell. Placing the new states back (if needed) requires the same number of inverse exchanges. This is a very modest number,  $s \sim 40$ , even for a rather large device of  $n \sim 10^3$  with Hilbert space of  $N = 2^n \sim 10^{300}$  dimensions. Hence, all basic operations of quantum computation can be performed on any physical system composed of non-holonomic triads of two-level subsystems arranged in a tree-like structure, and each operation can be completed within  $64 \times 16 \times 12 \times \log_3 n$  control intervals  $\tau$ . Note that the identity transformation should be applied to all other cells to preserve their states during the operation.

The second arrangement of cells (Fig. 3(b)) is mainly designed to emulate the dynamics of quantum fields on lattices. Of course, it can also perform general operations on any triad, but for a higher cost of  $s = O(n^{1/2})$ . In this arrangement, after each control period of  $64\tau$  the closest neighboring atoms are differently regrouped in triads (cells), with the original grouping repeating itself after three consecutive periods. Therefore, at each moment the change of the cell state depends on the states of the neighboring cells, as it should be in order to emulate the dynamics of the fields. Immediate analogy to the Ising model emerges when we restrict ourselves to small values of  $\epsilon$  for which terms of order  $\epsilon^2$  are negligible, and then each  $T = 64\tau$  period plays the role of the time increment  $\Delta\tau = \epsilon$ . The evolution of such a device is determined by three sums of effective cell Hamiltonians,  $\hat{H}_{eff}^{(p)} = \sum_q \hat{\mathcal{H}}_{q,p}$ , one for each period  $p = 1, 2, 3$ , where  $\hat{\mathcal{H}}_{q,p}$  is the effective Hamiltonian of the  $q$ th cell at the  $p$ th period.

We can cast the cell Hamiltonians to sums of tensor products of Pauli matrices  $\hat{\sigma}_\alpha^i$ , where the Greek index  $\alpha = x, y, z$  denotes the matrix type and the Latin index  $i$  specifies the two-level atom on which it acts. Since the cells are under complete control, the coefficients of this development can be made an arbitrary





function of the time  $\tau$ , and hence the effective Hamiltonian reads

$$\begin{aligned}\hat{H}_{eff}(\tau) = & A_i^\alpha(\tau)\hat{\sigma}_\alpha^i + B_{(i,j)}^{\alpha\beta}(\tau)\hat{\sigma}_\alpha^i\hat{\sigma}_\beta^j \\ & + C_{(i,j,k)}^{\alpha\beta\gamma}(\tau)\hat{\sigma}_\alpha^i\hat{\sigma}_\beta^j\hat{\sigma}_\gamma^k,\end{aligned}\quad (9)$$

with implicit summation over repeated indices, where  $(i, j)$  and  $(i, j, k)$  indicate pairs and triads of distinct atoms that are periodically grouped in a common cell. This Hamiltonian results in the evolution equation for the Heisenberg operators  $\hat{\sigma}_\alpha^i(\tau)$ ,

$$\begin{aligned}\hbar\frac{d\hat{\sigma}_\alpha^i(\tau)}{d\tau} = & \mathcal{A}_{\alpha,j}^{i,\beta}(\tau)\hat{\sigma}_\beta^j(\tau) + \mathcal{B}_{\alpha,(j,k)}^{i,\beta\gamma}(\tau)\hat{\sigma}_\beta^j(\tau)\hat{\sigma}_\gamma^k(\tau) \\ & + \mathcal{C}_{\alpha,(j,k,l)}^{i,\beta\gamma\delta}(\tau)\hat{\sigma}_\beta^j(\tau)\hat{\sigma}_\gamma^k(\tau)\hat{\sigma}_\delta^l(\tau),\end{aligned}\quad (10)$$

where the coefficients  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  are determined by  $A, B, C$  and the commutation relations of the Pauli matrices. By a proper choice of the coefficients  $A, B, C$  through the appropriate control sequences, one can simulate different linear and non-linear lattice models of quantum fields with time dependent parameters.

## 4 Toy Device

To conclude this paper, we now describe a toy device that can perform quantum computations on 9 qubits. An ensemble of 9 different Rydberg atoms, *i.e.* atoms of different elements or identical atoms which are excited to distinct pairs of Rydberg states, is placed in a magneto-optical trap at low temperature, as illustrated in Fig. 4. The best candidates for such a device are the long-living states corresponding to large angular momentum. By placing all the atoms in a static electric field, one lifts the degeneracy in the magnetic quantum number and performs tuning if needed. All the atoms experience the dipole-dipole interaction  $\hat{D}_{ij} = \hat{d}_i\hat{d}_j\langle R_{ij}^{-3} \rangle$ , where the cube of the inverse distance between atoms is averaged over their translational quantum states. Note, however, that only for almost resonant atoms this interaction is important. By a proper choice of the atomic states and the static field  $E_S$ , we obtain three triads,  $p = 1, 2, 3$ , each of which is composed of three almost resonant two-level atoms with transition frequencies centered on distinct frequencies  $\omega_p$ . For each triad  $p$ , the interactions  $\hat{D}_{ij}$  give the principal Hamiltonian, while a microwave field  $E_{\omega_p}$  at the frequency  $\omega_p$  serves as a control perturbation. Transportation of the state of one atom in each triad to the parent joint can be performed by dipole or Raman  $\pi$  transitions from the initial pair of Rydberg levels to a higher pair. With these higher pairs assumed nearly resonant with a frequency  $\omega_4$ , atoms 3, 6 and 7 form a higher-level triad—the parent joint of the first three triads—which is controlled by a fourth microwave field  $E_{\omega_4}$  of frequency  $\omega_4$ .

As an application of our technique, we show how to perform the discrete Fourier transform modulo  $N = 2^9 = 512$  on the toy device we have just pre-

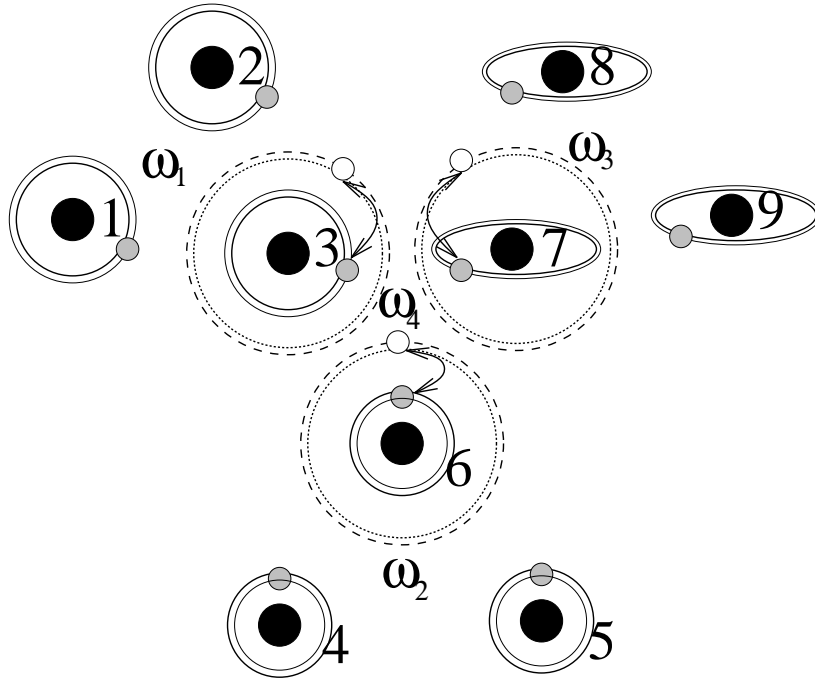


Figure 4: A toy device, composed of 9 Rydberg atoms, which can perform quantum computations on 9 qubits. Each atom is a two-level system shown schematically by double orbits. Atoms of different triads are excited to distinct pairs of Rydberg states. Each triad  $p$  is controlled by an external field of distinct frequency  $\omega_p$ . One atom in each triad can be excited to a pair of higher Rydberg states, thus forming a higher-level triad:  $(3, 6, 7)$ . These excitations (depicted by arrows) correspond to state transportations.

sented. This operation is achieved by the following 9-qubit unitary transformation

$$\hat{F}_N|x\rangle = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \exp(2\pi i xy/N) |y\rangle, \quad (11)$$

where  $|x\rangle$  and  $|y\rangle$  belong to the system computational basis, the states of which are defined by

$$|x\rangle \equiv |x_8\rangle_9 \dots |x_1\rangle_2 |x_0\rangle_1, \quad (12)$$

with  $x \equiv \sum_{r=0}^8 x_r 2^r = 0, 1, \dots, N-1$  ( $x_r = 0, 1$ ), and where  $|\cdot\rangle_i$  denotes the state of the  $i$ th atom—the  $i$ th qubit. The algorithm we employ to perform the Fourier transform is based on constructing the exponent in Eq. (11) as

$$\exp(2\pi i xy/2^9) = \prod_{r=0}^8 \prod_{s=0}^r \exp(i\pi x'_r y_s / 2^{r-s}), \quad (13)$$

where  $x'_r \equiv x_{8-r}$ . We begin by reversing the order in which the bits of the input  $x$  are stored in our 9-qubit register, that is, we achieve the unitary transformation

$$|x_8\rangle_9 \dots |x_1\rangle_2 |x_0\rangle_1 \rightarrow |x_0\rangle_9 \dots |x_7\rangle_2 |x_8\rangle_1 \quad (14)$$

by applying a sequence of state exchanges. Then we complete the transform in the following 9 steps: (i) We “split” the first qubit (the state of atom 1) by applying the unitary transformation

$$\hat{A} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \exp \left[ \frac{-i\pi}{\sqrt{8}} \begin{pmatrix} 1 - \sqrt{2} & 1 \\ 1 & -1 - \sqrt{2} \end{pmatrix} \right], \quad (15)$$

which maps  $|0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$  and  $|1\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ . Note that this would already complete the Fourier transform if we had only one qubit. (ii) Next, we apply to the first and second qubits the conditional phase shift  $|a\rangle_2 |b\rangle_1 \rightarrow e^{i\pi ab/2} |a\rangle_2 |b\rangle_1$  ( $a, b = 0, 1$ ), explicitly given by

$$\hat{B}_{21} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\pi/2} \end{pmatrix} = \hat{B}(\pi/2), \quad (16)$$

where  $\hat{B}(\phi)$  is the unitary transformation

$$\hat{B}(\phi) = \exp \left[ -i\phi \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right]. \quad (17)$$

Then we “split” the second qubit by applying the transformation  $\hat{A}$ . This accounts for the contribution of the second most significant bit of the input

$x$ . (iii) Similarly, in steps  $i = 3, 4, \dots, 9$  we apply the conditional phase shift  $|a\rangle_i |b\rangle_j \rightarrow e^{i\pi ab/2^{i-j}} |a\rangle_i |b\rangle_j$  ( $a, b = 0, 1$ ), that is,

$$\hat{B}_{ij} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\pi/2^{i-j}} \end{pmatrix} = \hat{B}(\pi/2^{i-j}), \quad (18)$$

to each pair of qubits  $(i, j)$ ,  $j = 1, 2, \dots, i-1$ , and then apply the transformation  $\hat{A}_i \equiv \hat{A}$  to the  $i$ th qubit. Note that after the  $i$ th step the first  $i$  qubits store the Fourier transform of the  $i$  most significant bits of  $x$ . Hence, after the 9th step the Fourier transform is completed:

$$\hat{F}_{2^9} = (\hat{A}_9 \hat{B}_{98} \cdots \hat{B}_{91}) \cdots (\hat{A}_3 \hat{B}_{32} \hat{B}_{31}) (\hat{A}_2 \hat{B}_{21}) (\hat{A}_1). \quad (19)$$

Performing these operations implies also application of state exchanges whenever one needs to transfer the states of atoms  $i$  and  $j$  to a common unit cell for processing. A list of control commands ( $\delta C_k$  sequences) corresponding to Eqs. (14) and (19) can be written straightforwardly.

## 5 Conclusion

Non-holonomic control is very relevant in the context of quantum computation, since it allows one to perform any unitary evolution of the system chosen as a computer, or, in other terms, it is a way to implement any quantum gate. Moreover, direct control over the entire system is not always necessary in order to perform computations on the information it contains : indeed, a completely controlled quantum device can be constructed as the smart arrangement of universally controlled unit cells. Two examples of such architectures have been provided in this paper, which are built in accordance with the computational task they are supposed to carry out. Moreover, as an example, a toy device has been considered, which is able to perform a concrete computation on 9 qubits.

## A Toffoli gate

The Toffoli-gate transformation is the unitary transformation on three qubits,

$$\hat{U}_{Toff} |x_2\rangle |x_1\rangle |x_0\rangle = |x_2\rangle |x_1\rangle |x_0 \text{ XOR } (x_1 \text{ AND } x_2)\rangle, \quad (20)$$

which corresponds to the three-bit classical logic gate,

$$\begin{aligned} x_2 &\rightarrow x'_2 = x_2 \\ x_1 &\rightarrow x'_1 = x_1 \\ x_0 &\rightarrow x'_0 = x_0 \text{ XOR } (x_1 \text{ AND } x_2), \end{aligned} \quad (21)$$

introduced by Toffoli as a universal gate for classical reversible computation. It acts as a permutation of the computational basis states,  $|x\rangle \equiv |x_2\rangle|x_1\rangle|x_0\rangle$ ,  $x \equiv \sum_{r=0}^2 x_r 2^r = 0, 1, \dots, 7$ , given by the unitary matrix

$$\hat{U}_{Toff} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \quad (22)$$

This matrix can be presented as

$$\hat{U}_{Toff} = \exp(-i\pi\hat{\mathcal{H}}_{Toff}), \quad (23)$$

with the (idempotent) Hermitian matrix

$$\hat{\mathcal{H}}_{Toff} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}. \quad (24)$$

In our control scheme the Toffoli-gate transformation can be effected on the unit cell by repeating 8 times the transformation  $\hat{U}_{\epsilon/8} \equiv \exp(-i\pi\hat{\mathcal{H}}_{Toff}/8)$  ( $\epsilon = \pi$ ), which is directly attainable:  $\hat{U}(t = 64T) = \hat{U}_{\epsilon/8}$  (see Fig. 2(d)).

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